COMPRESSED SENSING MRI WITH COMBINED SPARSIFYING TRANSFORMS AND SMOOTHED ℓ_0 NORM MINIMIZATION

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ABSTRACT

Undersampling the k-space is an efficient way to speed up the magnetic resonance imaging (MRI). Recently emerged compressed sensing MRI shows promising results. However, most of them only enforce the sparsity of images in single transform, e.g. total variation, wavelet, etc. In this paper, based on the principle of basis pursuit, we propose a new framework to combine sparsifying transforms in compressed sensing MRI. Each transform can efficiently represent specific feature that the other can not. This framework is implemented via the state-of-art smoothed ℓ_0 norm in overcomplete sparse decomposition. Simulation results demonstrate that the proposed method can improve image quality when comparing to single sparsifying transform.

Index Terms— Sparse decomposition, compressed sensing, MRI, medical imaging, multiscale transform

1. INTRODUCTION

Undersampling the k-space is a good way to speed up magnetic resonance imaging (MRI). However, this will violate the Nyquist sampling rule and results in artifacts. Recently emerged compressed sensing [1,2] provides a firm foundation to reconstruct signal from fewer measurements than the Nyquist sampling rule requests. For the signal $x \in \mathbb{R}^N$ that can be represented by K nonzero terms, the signal can be reconstructed exactly with overwhelming probability when the number of acquired data M satifies $M \geq Const \cdot K \cdot \log N$. For a compressible signal $x \in \mathbb{R}^N$, the reconstruction error of compressed sensing is proportional to the error of approximating image with K largest nonzero terms in specific sparsifying transform domain.

Lustig *et al* first proposed the basic mathematical model of compressed sensing MRI (CS-MRI) [3]. However, only one sparsifying transform is applied in his model. Diverse sparsifying transforms that can sparsely represent different types of features of MRI images are discussed in [3] and the recommended transform is 2D wavelet. But traditional 2D wavelet obtained by tensor products of 1D wavelets is good at isolating the point discontinuities, but fails in sparsely representing the curve-like image features[4,5]. To overcome this shortage, wavelet can be replaced by other geometric image transforms, curvelet[4] or contourlet[5,6], to sparsely represent curves. But they are not good at representing point-like image features.

Since each transform can only sparsely represent one type of features, a combination of them is a good choice. In this paper, we combine these sparsifying transforms to provide a overcomplete dictionaries. Our method is directly inspired by the principle of basis pursuit[7], which tries to find an optimal superposition of dictionary elements. For compressed sensing MRI, images are reconstructed from undersampled k-space data by searching the sparsest representation via ℓ_0 quasi-norm (it is not exactly a metric) minimization.

Compared with previous work, the advantages of our method are: (i) quality of reconstructed image is improved by enforcing its sparsity in combined sparsifying transforms; (ii) our combined transforms can avoid the bias of selected single transforms which may greatly reduce the image quality. We only consider noiseless measurements right now.

2. SMOOTHED ℓ_0 NORM IN CS-MRI

Compressed sensing is a new theory to reconstruct signal from undersampled measurements. Suppose that a noiseless signal $x \in \mathbb{C}^N$ is sampled by the sensing matrix $\Phi_{M\times N}$ ($M<N$), the measurements $y \in \mathbb{C}^M$ of x are

$$y_{M\times 1} = \Phi_{M\times N} x_{N\times 1}$$

Under the assumption that x can be sparsely represented in transform Ψ domain and α is the coefficient with respect to Ψ , x can be presented as

$$x = \Psi\alpha$$

CS tries to reconstruct the signal from undersampled measurements by minimizing ℓ_0 norm optimization. Let $\hat{\alpha}$ denote the estimation of α , the ℓ_0 norm optimization is

$$\hat{\alpha} = \min_{\alpha} \|\alpha\|_0 \quad s.t. \quad y = \Phi x$$

where $\|\alpha\|_0$ counts the non-zero elements of α.

In the field of MRI, the sampling matrix Φ is replaced by under-sampling Fourier operator F_u , which means partial Fourier coefficients are sensed. CS-MRI can be expressed as

$$\hat{\alpha} = \min_{\alpha} \|\alpha\|_0 \quad s.t. \quad y = F_u x \tag{1}$$

where y is the acquired k-space data and x be the image and Ψ be the dictionary.

Because ℓ_0 quasi-norm minimization is an NP hard problem, it is replaced by its closest linear counterpart, ℓ_1 norm minimization, and the results hold. However, the simpler solution is at the cost of increasing the number of required measurements for exact reconstruction [1,2,8].

A recent work proposed a relaxation that uses continuous function to approximate ℓ_0 quasi-norm [9] as

$$\|\alpha\|_0 \approx \mathrm{H}_\sigma(\alpha) = M - \sum_{i=1}^{M} f_\sigma(\alpha_i) = M - \sum_{i=1}^{M} \exp\left(\frac{-\alpha_i^2}{2\sigma^2}\right)$$

where α_i is the i^{th} element of vector α with length M .

When smoothed ℓ_0 is applied in CS-MRI, we can reconstruct the MR images from undersampled k-space by solving the following problem

$$\widehat{\alpha} = \arg\min_{\alpha} \mathrm{H}_\sigma(\alpha) \quad s.t. \quad y = F_u x$$

and the reconstructed image is $\hat{x} = \Psi\alpha$

Smoothed ℓ_0 minimization can reduce the required sampling rate to gain expected reconstruction quality with a given dictionary. However, it is limited to the sparsity of the image. This motivates us to construct a bigger and more expressive dictionary to enhance the sparsity of an image in it. In this paper, we extend the smoothed ℓ_0 approach to 2D compress sensing MRI and propose a method using combined sparsifying transforms.

3. COMBINED SPARSIFYING TRANSFORMS

As a major approach to solve CS, basis pursuit [7] suggests improving the sparsity of signal x with length N in overcomplete waveform dictionaries $\mathbf{\Psi}=[\Psi_1, \Psi_2, \cdots\Psi_j\cdots, \Psi_M]^T$ ($M>N$). Each waveform Ψ_j is a row vector with length N . Then coefficients $\alpha_{M\times 1} = \mathbf{\Psi}_{M\times N} x_{N\times 1}$ and each entry α_j is the inner product $\langle \psi_j, x\rangle$. MR image can be reconstructed from undersampled k-space data via finding solution to

$$\arg\min_{x} \|\alpha\|_1 \quad s.t. \quad y = F_u x \tag{2}$$

where $\|\alpha\|_1 = \sum_{j=1}^{M} |\alpha_j| = \sum_{j=1}^{M} |\langle \psi_j, x\rangle|$.

We view $\mathbf{\Psi}=[\Psi_1, \Psi_2, \cdots\Psi_j\cdots, \Psi_M]^T$ ($M>N$) as concatenation of the subsets $\{\psi_{\Lambda_i}, i = 1, 2, \cdots, I\}$, where Λ_i is the indices of waveforms in the ith subset.

$$\|\alpha\|_1 = \sum_{j=1}^{M} |\langle \psi_j, x\rangle| = \sum_{i=1}^{I} \sum_{j\in\Lambda_i} |\langle \psi_j, x\rangle| = \sum_{i=1}^{I} \|\langle \psi_{\Lambda_i}, x\rangle\|_1$$

This indicates that ℓ_1 norm minimization in global overcomplete dictionary is equivalent to minimize the sum of ℓ_1 norm of the dictionary's subsets. So, (2) can be written as

$$\arg\min_{x} \sum_{i=1}^{I} \|\langle \psi_{\Lambda_i}, x\rangle\|_1 \quad s.t. \quad y = F_u x$$

This provides us with an opportunity to recover the MR image from union of subsets of waveforms dictionary.

The model can be extended to ℓ_0 norm seamlessly. ℓ_0 norm minimization in global overcomplete dictionary is equivalent to minimize the sum of ℓ_0 norm of the dictionary's subsets. So, the combined sparsifying transforms for ℓ_0 norm can be expressed as

$$\arg\min_{x} \sum_{j=1}^{M} \|\Psi_j{}^T x\|_0 \quad s.t. \quad y = F_u x \tag{3}$$

In this paper, we consider the condition each sub-dictionary Ψ_j comes from commonly used transforms which sparsify different types of image features. Unfortunately, the matrix of dictionary Ψ and its adjoint Ψ^T are rarely explicitly constructed in memory, e.g. curvelet and contourlet. Instead, they are implemented as fast implicit analysis and synthesis operators. Let T and T* denotes the operator pair, $\Psi^T x = Tx$ and $T^* T = I$. So that even if Ψ is a tight frame and it may not be orthogonal, i.e. $\Psi\Psi^T = cI, c > 0$, we don't have to normalize T and T*[10].

Furthermore, storing and computation of α is expensive because dimension of α is higher than dimension of signal x for overcomplete dictionary. So we apply fast forward transform T_i on the image and get another version of (3)

$$\arg\min_{x} \sum_{i=1}^{I} \|T_i x\|_0 \quad s.t. \quad y = F_u x \tag{4}$$

This model assumes each transform is regularized, i.e. that the ℓ_2 norm of the dictionary's each column equals to 1. In practice, the condition is more complicated. An effective solution to solve this problem is to enforce the sparsity in each transform domain individually.

For the transforms under consideration, an ideal case is to construct an infinite dictionary that contains any possible waveform. In this case, if a waveform exactly the same with target image is included, such that the image we are to reconstruct can be represented with only one coefficient. However, this case is obviously unrealistic. Instead, we try to find several complement transforms, in other words, the coherence between them is expected to be as low as possible, namely the uncertainty principle. The Morphological Component Analysis (MCA)[10] framework assumes an image x can be composed into several components, each component x_j can be sparsely represented in an associated basis Ψ_j. For each j, the representation of x_j

in Ψ_j is sparsest, and in any other Ψ_i $(i \neq j)$, it is not or at least not as sparse as another sparsifying transform.

Considering talents in representing different image features and the computing complexity[5,6], we adopted wavelet and an improved contourlet[6] to represent point-like and curve-like image features respectively.

4. IMPLEMENTATION OF THE COMBINED APPROACH

In order to solve equation (4), we extend sigma annealing method proposed in [9] to compressed sensing MRI with combined sparsifying transforms.

Without loss of generality, suppose we have K different sparsifying transforms combined. When minimizing the coefficients in the transform domain of T_k, which consist in the k^{th} subset of the global overcomplete dictionary, we assume the coefficients corresponding to other transform domains are fixed. Theoretically, the order of transforms is arbitrary as long as we traverse all the selected transforms in $\{T_k \mid k = 1,2,\cdots,K\}$. During the calculation process, we keep projecting the coefficients back onto the feasible set so that the data consistency is ensured.

Given an annealing sequence of $[\sigma_1, \sigma_2, \cdots, \sigma_J]$, we minimize smoothed ℓ_0 norm iteratively for each σ with general gradient descent method. In implementation of smoothed ℓ_0 norm, value of σ at $j+1$ iteration is $\sigma_{j+1} = d * \sigma_j$ ($0 < d < 1$). A decreasing sequence of sigma is carefully selected and the minimization of smoothed ℓ_0 norm is roughly calculated for each sigma in the sequence. Simulations in [9] demonstrate the smoothed ℓ_0 norm algorithm converges fast. In [8] [4], the authors proposed to solve the Lagrange form of (1). But as mentioned before, the equivalent dictionary of some geometric transform's fast algorithm is not regularized, i.e. the ℓ_2 norm of each column is not equal to 1. So we had better force the sparsity of image in each transform domain individually and keep projecting the results back to feasible set after each optimization step.

The reconstruction algorithm with smoothed ℓ_0 norm and combined sparsifying transforms (SL0-CST) is as follows

Algorithm 1 Reconstruction algorithm with smoothed ℓ_0 norm and combined sparsifying transforms

Initialization:

1) Let x_0 be the minimum ℓ_2 norm solution of $y = A\alpha$ obtained by pseudo-inverse of A.

2) Choose a suitable annealing strategy for σ and get its sequence $[\sigma_1, \sigma_2, \cdots, \sigma_j, \cdots, \sigma_J]$.

Main Loop:

for j=1,2,…,J

  Set $\sigma = \sigma_j$, $x = \hat{x}_j$

  for k = 1,2,…,K, where K is the number of transforms employed..

    1) $\alpha_k = \psi_k x$;

    2) $g_k = \alpha_k \exp\left(-\frac{\alpha_k}{2\sigma^2}\right)$

    3) $x = x - t \cdot \psi_k g_k$ with step length t

    4) $x = x - \lambda F_u (F_u^* x - y)$ where λ controls the data consistency and $\lambda = 1$ by default.

  end for

  Set $\hat{x}_j = x$

end for

Final answer $\hat{x} = \hat{x}_J$

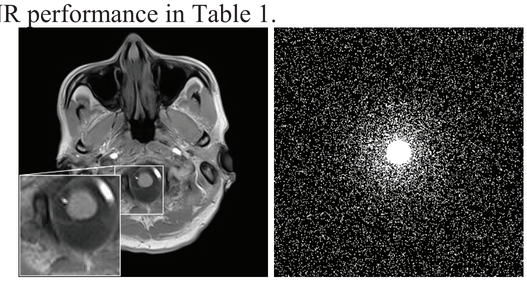
5. SIMULATION RESULTS

To validate the performance, the proposed method is compared with the nonlinear conjugate (CG) ℓ_1 norm when single and combined sparsifying transforms are employed [11]. Parameters of nonlinear conjugate ℓ_1 norm are the same as [11]. Parameters of smoothed ℓ_0 norm are set as decay factor d of σ is 0.5 and the stop criteria, smallest σ, is 10^{-4}. The limitations of outer and inner iterations are 50 and 5. Besides the visual appearance, power signal-to-noise ratio (PSNR) is served as objective criteria and defined as

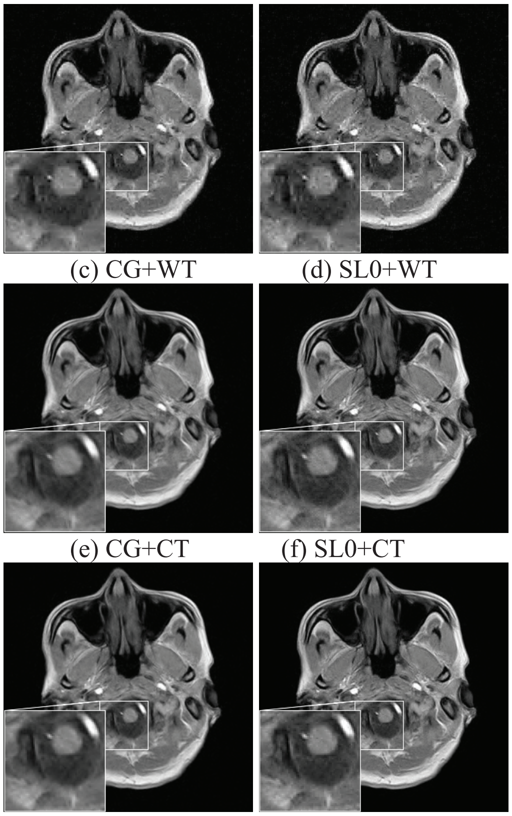
$$PSNR = 20\log_{10}\left(\frac{255}{\sqrt{MSE}}\right)$$

where $MSE = \frac{1}{P \times Q}\sum_{p=1}^{P}\sum_{q=1}^{Q}\left(\tilde{x}(p,q) - \hat{x}(p,q)\right)^2$ and (p,q) is pixel location of a $P \times Q$ image. $\hat{x}$ is the reconstructed image and $\tilde{x}$ is reconstructed image from fully sampled k-space data. PSNR evaluates the difference in gray values between $\hat{x}$ and $\tilde{x}$.

We adopt the variable density sampling pattern, shown in Fig.1 (b), to acquire 15% of k-space data. The reconstructed curves via contourlet are much clearer than those of wavelet. No matter in the ℓ_1 norm or ℓ_0 norm, combined sparsifying transforms can improve the image quality than single transform. Due to the approximation of ℓ_0 norm, smoothed ℓ_0 norm with combined sparsifying transforms obtains the best results. This conclusion is in accordance with the PSNR performance in Table 1.



(a) original image (b) sampling mask



(c) CG+WT (d) SL0+WT

(e) CG+CT (f) SL0+CT

(g) CG+WT+CT (h) SL0+WT+CT

Figure 1. Comparisons on reconstructed images. CG denotes nonlinear conjugate ℓ_1 norm and SL0 denotes smoothed ℓ_0 norm, CT means contourlet and WT means wavelet.

Table 1. PSNR performance

Types of norm	Sparsifying transforms	PSNR
Nonlinear conjugate ℓ_1	Wavelet	29.02
	Contourlet	31.81
	Wavelet+Contourlet	32.56
Smoothed ℓ_0	Wavelet	28.16
	Contourlet	32.36
	Wavelet+Contourlet	34.23

6. CONCLUSION

The combined sparsifying transforms for smooth ℓ_0 norm minimization is proposed to reconstruct the magnetic resonance images from noiseless undersampled k-space data. Theoretical analysis and simulation results demonstrate that the proposed method can improve image quality than single sparsifying transform. However, the robust to noisy k-space data is the future work.

7. ACKNOWLEDGEMENT

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This work was partially supported by NNSF of China under Grants (10774125,10605019 and 10974164). Thank Mohimani H., Lustig M., Lu Y. and Trzasko J. for sharing their code.

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